

On the Motion of a Free Particle in the de Sitter Manifold

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Abstract

Let $M = SO(1, 4)/SO(1, 3) \simeq S^3 \times \mathbb{R}$ (a parallelizable manifold) be a submanifold in the structure $(\dot{M}, \dot{\mathbf{g}})$ (hereafter called the bulk) where $\dot{M} \simeq \mathbb{R}^5$ and $\dot{\mathbf{g}}$ is a pseudo Euclidian metric of signature $(1, 4)$. Let $\mathbf{i} : M \rightarrow \mathbb{R}^5$ be the inclusion map and let $\mathbf{g} = \mathbf{i}^* \dot{\mathbf{g}}$ be the pullback metric on M . It has signature $(1, 3)$. Let \mathbf{D} be the Levi-Civita connection of \mathbf{g} . We call the structure (M, \mathbf{g}) a de Sitter manifold and $M^{dSL} = (M = \mathbb{R} \times S^3, \mathbf{g}, \mathbf{D}, \tau_{\mathbf{g}}, \uparrow)$ a de Sitter spacetime structure, which is of course orientable by $\tau_{\mathbf{g}} \in \sec \bigwedge^4 T^*M$ and time orientable (by \uparrow). Under these conditions we prove that if the motion of a free particle moving on M happens with constant *bulk* angular momentum then its motion in the structure M^{dSL} is a timelike geodesic. Also any geodesic motion in the structure M^{dSL} implies that the particle has constant angular momentum in the bulk.

1 Introduction

In what follows $SO(1, 4)$ and $SO(1, 3)$ denote the special pseudo-orthogonal groups in $\mathbb{R}^{1,4} = (\dot{M} = \mathbb{R}^5, \dot{\mathbf{g}})$ where $\dot{\mathbf{g}}$ is a metric of signature $(1, 4)$. The *de Sitter manifold* M can be viewed as a brane (a submanifold) in the structure $\mathbb{R}^{1,4}$. The structure $M^{dSL} = (M, \mathbf{g}, \mathbf{D}, \tau_{\mathbf{g}}, \uparrow)$ will be called *Lorentzian de Sitter spacetime structure* where, if $\iota : \mathbb{R} \times S^3 \rightarrow \mathbb{R}^5$ is the inclusion mapping, $\mathbf{g} = \iota^* \dot{\mathbf{g}}$ and \mathbf{D} is the parallel projection on M of the pseudo Euclidian metric compatible connection $\dot{\mathbf{D}}$ in $\mathbb{R}^{1,4}$ (details in [7, 8]). As well known, (M, \mathbf{g}) , a pseudo-sphere is a spacetime of constant Riemannian curvature. It has ten Killing vector fields. The Killing vector fields are the generators of infinitesimal actions of the group $SO(1, 4)$ (called the de Sitter group) in M . The group $SO(1, 4)$ acts transitively¹

¹A group G of transformations in a manifold M ($\sigma : G \times M \rightarrow M$ by $(g, x) \mapsto \sigma(g, x)$) is said to act transitively on M if for arbitrariness $x, y \in M$ there exists $g \in G$ such that $\sigma(g, x) = y$.

in $SO(1, 4)/SO(1, 3)$, which is thus a homogeneous space (for $SO(1, 4)$).

The structure M^{dSL} has been used by many physicists as an alternative arena for the motion of particles and fields in place of the Minkowski spacetime structure² \mathfrak{M} . One of the reasons is that the isometry group of the structure (M, g) is the de Sitter group, which as well known reduces to the Poincaré group when the radius ℓ of (M, g) goes to ∞ . Now, as well known the natural motion of a free particle of mass m in \mathfrak{M} occurs with constant momentum $\mathbf{p} = m\kappa_*$ where $\kappa : \mathbb{R} \rightarrow \mathcal{M}$ is a timelike curve pointing to the future. The question which naturally arises is the following:

Which is the natural motion of a free particle of mass m in the structure (M, g) ?

One natural suggestion given the well known relation between the de Sitter and Poincaré groups [2] is that such a motion occurs with constant angular momentum \mathbf{L} as determined by (hyper observers) living in the bulk. Given this hypothesis we prove the following proposition: (a): If a particle travels with geodesic motion in the structure M^{dSL} then its bulk angular momentum \mathbf{L} is constant. (b): Also, if the motion of a particle of mass m constrained to live in M occurs with constant bulk angular \mathbf{L} then its motion for an observer living in the brane M is described by a timelike geodesic in the structure M^{dSL} .

The paper is composed of three sections. Section 2, called preliminaries fixes the necessary notations and describes the geodesic equation of motion (of a free particle of mass m) in the structure M^{dSL} and the equation of motion in that structure that must be valid if a particle of mass m moves in M with constant bulk angular momentum \mathbf{L} . In Section 3 we present a proof of the proposition referred above and finally in Section 4 we present our conclusions.

2 Preliminaries

We now recall the description of the manifold $\mathbb{R} \times S^3$ as a pseudo-sphere (a submanifold) of radius ℓ of the in $\dot{M} = \mathbb{R}^5$. If $(X^0, X^1, X^2, X^3, X^4)$ are the global Euclidian coordinates of $\dot{M} = \mathbb{R}^5$ then the equation representing the pseudo sphere is

$$(X^0)^2 - (X^1)^2 - (X^2)^2 - (X^3)^2 - (X^4)^2 = -\ell^2. \quad (1)$$

Introducing *conformal coordinate functions* $\{\mathbf{x}^\mu\}$ for M by projecting the points of $\mathbb{R} \times S^3$ from the “north-pole” to a plane tangent to the “south pole” we see immediately that the coordinates³ $\{x^\mu\}$ covers all $\mathbb{R} \times S^3$ except the “north-pole”. We immediately find that⁴

²Minkowski spacetime is the structure $\mathfrak{M} = (\mathcal{M} = \mathbb{R}^4, \boldsymbol{\eta}, D, \tau_{\boldsymbol{\eta}}, \uparrow)$ where $\boldsymbol{\eta}$ is the usual Minkowski metric, $\tau_{\boldsymbol{\eta}} \in \sec \wedge^4 T^* \mathcal{M}$ defines an orientation and \uparrow denotes that $(\mathcal{M}, \boldsymbol{\eta})$ is time orientable. Details in [6].

³We denote the coordinates of a point $p \in M$ covered by the coordinate functions \mathbf{x}^μ by $x^\mu = \mathbf{x}^\mu(p)$.

⁴The matrix with entries $\eta_{\mu\nu}$ is the diagonal matrix $\text{diag}(1, -1, -1, -1)$.

$$\mathbf{g} = i^* \mathring{\mathbf{g}} = \Omega^2 \eta_{\mu\nu} dx^\mu \otimes dx^\nu \quad (2)$$

where

$$X^\mu = \Omega x^\mu, \quad X^4 = -\ell \Omega \left(1 + \frac{\sigma^2}{4\ell^2} \right) \quad (3)$$

$$\Omega = \left(1 - \frac{\sigma^2}{4\ell^2} \right)^{-1} \quad (4)$$

and

$$\sigma^2 = \eta_{\mu\nu} x^\mu x^\nu := x_\mu x^\mu. \quad (5)$$

Now, writing $\mathbf{D}_{\partial_\mu} \partial_\nu = \Gamma_{\mu\nu}^{\alpha} \partial_\alpha$ the non null connection coefficients are:

$$\begin{aligned} \Gamma_{00}^0 &= \frac{\Omega}{2\ell^2} x^0, \quad \Gamma_{01}^0 = -\frac{\Omega}{2\ell^2} x^1, \quad \Gamma_{02}^0 = -\frac{\Omega}{2\ell^2} x^2, \quad \Gamma_{03}^0 = -\frac{\Omega}{2\ell^2} x^3, \quad \Gamma_{11}^0 = \frac{\Omega}{2\ell^2} x^0, \quad \Gamma_{22}^0 = \frac{\Omega}{2\ell^2} x^0, \quad \Gamma_{33}^0 = \frac{\Omega}{2\ell^2} x^0, \\ \Gamma_{00}^1 &= -\frac{\Omega}{2\ell^2} x^1, \quad \Gamma_{01}^1 = \frac{\Omega}{2\ell^2} x^0, \quad \Gamma_{11}^1 = -\frac{\Omega}{2\ell^2} x^1, \quad \Gamma_{12}^1 = -\frac{\Omega}{2\ell^2} x^2, \quad \Gamma_{13}^1 = -\frac{\Omega}{2\ell^2} x^3, \quad \Gamma_{22}^1 = \frac{\Omega}{2\ell^2} x^1, \quad \Gamma_{33}^1 = \frac{\Omega}{2\ell^2} x^1, \\ \Gamma_{00}^2 &= -\frac{\Omega}{2\ell^2} x^2, \quad \Gamma_{02}^2 = \frac{\Omega}{2\ell^2} x^0, \quad \Gamma_{11}^2 = \frac{\Omega}{2\ell^2} x^2, \quad \Gamma_{12}^2 = -\frac{\Omega}{2\ell^2} x^1, \quad \Gamma_{22}^2 = -\frac{\Omega}{2\ell^2} x^2, \quad \Gamma_{23}^2 = -\frac{\Omega}{2\ell^2} x^3, \quad \Gamma_{33}^2 = \frac{\Omega}{2\ell^2} x^2, \\ \Gamma_{00}^3 &= -\frac{\Omega}{2\ell^2} x^3, \quad \Gamma_{03}^3 = \frac{\Omega}{2\ell^2} x^0, \quad \Gamma_{11}^3 = \frac{\Omega}{2\ell^2} x^3, \quad \Gamma_{13}^3 = -\frac{\Omega}{2\ell^2} x^1, \quad \Gamma_{22}^3 = \frac{\Omega}{2\ell^2} x^3, \quad \Gamma_{23}^3 = -\frac{\Omega}{2\ell^2} x^2, \quad \Gamma_{33}^3 = -\frac{\Omega}{2\ell^2} x^3. \end{aligned} \quad (6)$$

Let $\sigma : I \rightarrow M, s \mapsto \sigma(s)$ be a time like geodesic in M . Its tangent vector field σ_* such that $\sigma_*(s) = \frac{d\sigma \circ \sigma(s)}{ds} \frac{\partial}{\partial x^\mu} \Big|_\sigma = \frac{dx^\mu}{ds} \frac{\partial}{\partial x^\mu}$ satisfy $\mathbf{D}_{\sigma_*} \sigma_* = 0$ and in components it is

$$\frac{d^2 x^\alpha}{ds^2} + \Gamma_{\mu\nu}^\alpha \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = 0. \quad (7)$$

Using the connection coefficients given by Eq.(6) Eq.(7) becomes

$$\frac{d^2 x^\alpha}{ds^2} + \frac{\Omega}{\ell^2} x_\mu \frac{dx^\mu}{ds} \frac{dx^0}{ds} - \frac{\Omega}{2\ell^2} x^0 \frac{dx_\mu}{ds} \frac{dx^\mu}{ds} = 0. \quad (8)$$

2.1 Equations of Motion in M from Constant Angular Momentum in the Bulk

Let $\{\mathbf{E}_A = \frac{\partial}{\partial X^A}\}$, $A = 0, 1, 2, 3, 4$ be the canonical basis of $T\mathring{M} = T\mathbb{R}^5$ and let $\{E^A = dX^A\}$ be a basis of $T^*\mathring{M}$ dual to $\{\mathbf{E}_A = \frac{\partial}{\partial X^A}\}$. We have

$$\mathring{\mathbf{g}} = \boldsymbol{\eta}_{AB} E^A \otimes E^B \quad (9)$$

where the matrix with entries $\boldsymbol{\eta}_{AB}$ is the diagonal matrix $\text{diag}(1, -1, -1, -1, -1)$. Moreover let $\mathring{\mathbf{g}} = \eta^{AB} \mathbf{E}_A \otimes \mathbf{E}_B$ be the metric of the cotangent bundle (with $\eta^{AC} \boldsymbol{\eta}_{CB} = \delta_B^A$). Finally let $\{E_A\}$ be the reciprocal basis of $\{E^A\}$, i.e., $\mathring{\mathbf{g}}(E^A, E_B) = \delta_B^A$. We introduce the basis $\{\mathcal{E}_A\}$ of \mathbb{R}^5 and make the usual identification $\mathbf{E}_A(p) \simeq \mathbf{E}_A(p') = \mathfrak{E}_A$, $E_A(p) \simeq E_A(p') = \mathcal{E}_A$ for any $p, p' \in \mathbb{R}^5$.

Let $\mathbf{X} = X^A \mathcal{E}_A$ be the position covector, $\mathbf{P} = m \ddot{X}^B \mathcal{E}_B$ the bulk momentum covector and $\mathbf{L} = \mathbf{X} \wedge \mathbf{P}$ the bulk angular momentum of a particle of mass m in the bulk spacetime $\mathbb{R}^{1,4}$. If the particle is constrained to move "freely"⁵ in the submanifold $\mathbb{R} \times S^3$ a natural hypothesis is that its bulk angular momentum is a constant of motion. Now, $\mathbf{L} = \text{cte}$ implies immediately

$$\frac{1}{2}(X^A \ddot{X}^B - \ddot{X}^A X^B) \mathcal{E}_A \wedge \mathcal{E}_B = 0. \quad (10)$$

Thus, for $\kappa, \iota = 0, 1, 2, 3$ it is $X^\kappa \ddot{X}^\iota - \ddot{X}^\kappa X^\iota = 0$, i.e.,

$$x^k \left(\frac{dx^i}{ds} \frac{1}{\ell^2} \Omega^2 x_i \frac{dx^l}{ds} + \Omega \frac{d^2 x^l}{ds^2} \right) - \left(\frac{dx^i}{ds} \frac{1}{\ell^2} \Omega^2 x_i \frac{dx^k}{ds} + \Omega \frac{d^2 x^k}{ds^2} \right) x^l = 0 \quad (11)$$

which is the equation of motion according to the structure M^{dSL} .

Also, from the equation $X^\mu \ddot{X}^4 - \ddot{X}^\mu X^4 = 0$ we get

$$\begin{aligned} & -(2\Omega - 1) \left(\frac{d^2 x^b}{ds^2} + \frac{1}{\ell^2} \Omega x_i \frac{dx^i}{ds} \frac{dx^b}{ds} \right) \\ & + \frac{1}{2\ell^2} \Omega \left(\frac{1}{\ell^2} \Omega x_i x_j \frac{dx^i}{ds} \frac{dx^j}{ds} + \frac{dx_i}{ds} \frac{dx^i}{ds} + x_i \frac{d^2 x^i}{ds^2} \right) x^b = 0. \end{aligned} \quad (12)$$

3 Constant Bulk Angular Momentum versus Geodesic Equation

We have the following

Proposition 1 (a): *If a particle travels with geodesic motion in the structure M^{dSL} then its bulk angular momentum \mathbf{L} is constant.* (b): *Also, if the motion of a particle of mass m constrained to move in M occurs with constant bulk angular \mathbf{L} then its motion for an observer living in the brane M is described by a timelike geodesic in the structure M^{dSL} .*

Proof: (a1) We multiply the geodesic equation for component $x^\kappa(s)$ by $x^\iota(s)$ and the geodesic equation for component $x^\iota(s)$ by $x^\kappa(s)$ thus getting:

$$\frac{d^2 x^\kappa}{ds^2} x^\iota + \frac{\Omega}{\ell^2} x_\mu \frac{dx^\mu}{ds} \frac{dx^\kappa}{ds} x^\iota - \frac{\Omega}{2\ell^2} x^\kappa \frac{dx_\mu}{ds} \frac{dx^\mu}{ds} x^\iota = 0, \quad (13)$$

$$x^\kappa \frac{d^2 x^\iota}{ds^2} + x^\kappa \frac{\Omega}{\ell^2} x_\mu \frac{dx^\mu}{ds} \frac{dx^\iota}{ds} - x^\kappa \frac{\Omega}{2\ell^2} x^\iota \frac{dx_\mu}{ds} \frac{dx^\mu}{ds} = 0. \quad (14)$$

and subtracting Eq.(14) from Eq.(13) we get

$$x^\kappa \left(\frac{d^2 x^\iota}{ds^2} + \frac{\Omega}{\ell^2} x_\mu \frac{dx^\mu}{ds} \frac{dx^\iota}{ds} \right) - \left(\frac{d^2 x^\kappa}{ds^2} + \frac{\Omega}{\ell^2} x_\mu \frac{dx^\mu}{ds} \frac{dx^\kappa}{ds} \right) x^\iota = 0. \quad (15)$$

⁵From a physical point of view the statement moving 'freely' means that observers living in M cannot detect any force acting on the particle.

Multiplying Eq.(15) by Ω we get Eq.(11) which is the equation of motion for the particle coming from the hypothesis that its bulk angular momentum is constant.

(a2) From the geodesic equation (Eq.(8)) we easily have the following two equations

$$\begin{aligned} (2\Omega - 1)\frac{d^2 x^k}{ds^2} + (2\Omega - 1)\frac{\Omega}{\ell^2}x_i\frac{dx^i}{ds}\frac{dx^k}{ds} - (2\Omega - 1)\frac{\Omega}{2\ell^2}x^k\frac{dx_i}{ds}\frac{dx^i}{ds} &= 0, \\ -\frac{1}{2\ell^2}\Omega x_i\frac{d^2 x^i}{ds^2}x^k - \frac{1}{2\ell^2}\Omega\frac{\Omega}{\ell^2}x_i x_j\frac{dx^j}{ds}\frac{dx^i}{ds}x^k + \frac{1}{2\ell^2}\Omega\frac{\Omega}{2^2}x_i x^i\frac{dx_j}{ds}\frac{dx^j}{ds}x^k &= 0 \end{aligned} \quad (16)$$

and thus summing these equations we get:

$$\begin{aligned} (2\Omega - 1)\left(\frac{d^2 x^k}{ds^2}\right) + (2\Omega - 1)\frac{\Omega}{\ell^2}x_i\frac{dx^i}{ds}\frac{dx^k}{ds} - \frac{1}{2\ell^4}\Omega^2 x_i x_j x^k\frac{dx^i}{ds}\frac{dx^j}{ds} \\ - \frac{\Omega}{2\ell^2}x^k\frac{dx_i}{ds}\frac{dx^i}{ds} - \frac{1}{2\ell^2}\Omega x_i\left(\frac{d^2 x^i}{ds^2}\right)x^k &= 0 \end{aligned} \quad (17)$$

Comparison of Eq.(17) with Eq.(12). plus the result obtained in (a1) proves that geodesic motion in the structure M^{dSL} implies motion with constant angular momentum in the bulk.

(b) Let us show now that constant bulk angular momentum \mathbf{L} implies in geodesic motion in the structure M^{dSL} .

We already know that the equations of motion coming from the hypothesis that $\mathbf{L} = \mathbf{cte}$ is for $\alpha, \beta = 0, 1, 2, 3$

$$\begin{aligned} X^\alpha \ddot{X}^\beta - \ddot{X}^\alpha X^\beta &= 0 \quad \alpha, \beta = 0, 1, 2, 3, \\ X^\beta \ddot{X}^4 - \ddot{X}^\beta X^4 &= 0, \end{aligned}$$

which can be written respectively using Eqs. (3), (4) and (5) as

$$x^\alpha \left(\frac{dx^\mu}{ds} \frac{1}{\ell^2} \Omega^2 x_\mu \frac{dx^\beta}{ds} + \Omega \frac{d^2 x^\beta}{ds^2} \right) - \left(\frac{dx^\mu}{ds} \frac{1}{\ell^2} \Omega^2 x_\mu \frac{dx^\alpha}{ds} + \Omega \frac{d^2 x^\alpha}{ds^2} \right) x^\beta = 0 \quad (18)$$

and

$$\begin{aligned} -(2\Omega - 1) \left(\frac{d^2 x^\beta}{ds^2} + \frac{1}{\ell^2} \Omega x_\mu \frac{dx^\mu}{ds} \frac{dx^\beta}{ds} \right) \\ + \frac{1}{2\ell^2} \Omega \left(\frac{1}{\ell^2} \Omega x_\mu x_\nu \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} + \frac{dx_\mu}{ds} \frac{dx^\mu}{ds} + x_\mu \frac{d^2 x^\mu}{ds^2} \right) x^\beta &= 0 \end{aligned} \quad (19)$$

Multiplying Eq.(18) by x_α (and summing in α) we get

$$x_\alpha x^\alpha \left(\frac{dx^\mu}{ds} \frac{1}{\ell^2} \Omega^2 x_\mu \frac{dx^\beta}{ds} + \Omega \frac{d^2 x^\beta}{ds^2} \right) - \left(\frac{dx^\mu}{ds} \frac{1}{\ell^2} \Omega^2 x_\mu x_\alpha \frac{dx^\alpha}{ds} + \Omega x_\alpha \frac{d^2 x^\alpha}{ds^2} \right) x^\beta = 0 \quad (20)$$

which can be written as

$$\frac{1}{4\ell^2}\Omega\sigma^2\left(\frac{dx^\mu}{ds}\frac{1}{\ell^2}\Omega x_\mu\frac{dx^\beta}{ds}+\frac{d^2x^\beta}{ds^2}\right)-\frac{1}{4\ell^2}\Omega\left(\frac{1}{\ell^2}\Omega x_\mu x_\alpha\frac{dx^\mu}{ds}\frac{dx^\alpha}{ds}+x_\alpha\frac{d^2x^\alpha}{ds^2}\right)x^\beta=0 \quad (21)$$

Summing Eq.(21) with Eq.(19) we get in sequence

$$\left(\frac{1}{2\ell^2}\Omega\sigma^2-(2\Omega-1)\right)\left(\frac{1}{\ell^2}\Omega x_\mu\frac{dx^\mu}{ds}\frac{dx^\beta}{ds}+\frac{d^2x^\beta}{ds^2}\right)+\frac{1}{2\ell^2}\Omega\frac{dx_\mu}{ds}\frac{dx^\mu}{ds}x^\beta=0,$$

$$-\left(\frac{1}{\ell^2}\Omega x_\mu\frac{dx^\mu}{ds}\frac{dx^\beta}{ds}+\frac{d^2x^\beta}{ds^2}\right)+\frac{1}{2\ell^2}\Omega\frac{dx_\mu}{ds}\frac{dx^\mu}{ds}x^\beta=0,$$

and finally $\frac{d^2x^\beta}{ds^2}+\frac{1}{\ell^2}\Omega x_\mu\frac{dx^\mu}{ds}\frac{dx^\beta}{ds}-\frac{1}{2\ell^2}\Omega x^\beta\frac{dx_\mu}{ds}\frac{dx^\mu}{ds}$. which is just the geodesic equation (see Eq.(8)) in the structure M^{dSL} . ■

4 Conclusions

We said in the introduction that the de Sitter structure M^{dSL} has been studied by many authors as a possible natural arena for the motion of particles and fields instead of the Minkowski spacetime structure \mathfrak{M} . In particular papers [4, 5] are devoted to study which could be the natural motion of particles in M^{dSL} . Authors of that papers claim that the natural path of free particles in M^{dSL} cannot be timelike geodesics in that structure and they build a new theory for finding the true geodesics for the structure (M, \mathbf{g}) . Their main argument of such a claim is that they became confused by a quotation in [3] saying that there are points in M which cannot be joined by a geodesic. However, the quotation in [3] is only partially valid. Indeed, it has been long showed in [10] that points that cannot be joined by a geodesic can only be joined by a spacelike curve, which of course cannot be the paths of any material particle. We discussed these issues at length in [8] where in particular it is shown also that the equations of motion found in [4, 5] are equivocated. At least we want to emphasize that recently it has been shown in [9] by using the Clifford and spin-Clifford formalisms [6] that the hypothesis that a particle moving freely in (M, \mathbf{g}) has constant bulk angular momentum leads naturally to the Dirac equation as found in [1] in the de Sitter structure (M, \mathbf{g}) .

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